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On the Behavior of Solutions of a Class of Nonlinear Partial Differential Equations

H. I. Abdel-Gawad[1]

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The behavior of the steady-state (or the traveling wave) solutions for a class of nonlinear partial differential equations is studied. The nonlinearity in these equations is expressed by the presence of the convective term. It is shown that the steady-state (or the traveling wave) solution may explode at a finite value of the spatial (or the characteristic) variable. This holds whatever the order of the spatial derivative term in these equations. Furthermore, new special solutions of a set of equations in this class are also found.

KEY WORDS: Qualitative behaviors; a class of nonlinear partial differential equations.

1. INTRODUCTION

We shall study the behavior of the solutions for a class of nonlinear partial differential equations (NLPDE). These equations are characterized by the presence of the convective term, the time derivative term and the spatial derivative terms. The spatial derivative term may be diffuse, dissipative or a linear combination of all orders. We model this class of NLPDE by

$$u_t + \lambda u u_x + F(u_{xx}, u_{xxx}, u_{xxxx}, \dots) = 0 \tag{1}$$

where $F(v, w, q, \dots)$ is a linear functional in the arguments. We shall study special cases of Eq. (1), namely

(i) When $F = \mu u_{xx}$, Eq. (1) becomes

$$u_t + \lambda u u_x + \mu u_{xx} = 0 \tag{2}$$

[1] Department of Mathematics, Faculty of Science, Cairo University, Cairo, Egypt.

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This equation had been introduced by Burgers[1] to model the viscosity term in the Navier–Stokes equations by μu_{xx}.

(ii) When $F=\mu u_{xxx}$, Eq. (1) becomes

$$u_t+\lambda u u_x+\mu u_{xxx}=0 \tag{3}$$

which is the Korteweg de Vries KdV equation. This equation describes the propagation of waves in long channels.

(iii) When $F=\mu u_{xx}+\nu u_{xxxx}$, Eq. (1) becomes

$$u_t+\lambda u u_x+\mu u_{xx}+\mu u_{xxxx}=0 \tag{4}$$

This is the Kuramoto–Sivashinsky KS equation. It describes the fluctuations in the position of a front of a flame[2] or the evolution of a homogeneous medium unstable against a spatially uniform oscillating chemical reactions.[3] Details on this equation may be found in ref. 4.

We shall present also a technique for finding some special solutions of Eqs. (1)–(4). This technique has a common property with the method of using the Painlevé analysis and auto-Bäklund transformation for fining some exact solutions of Eq. (1).[5–8] This common property is that we should have an overdetermined system of equations. First, we clarify this in brief for the Painlevé and auto-Bäklund analysis. The Painlevé test assumes that there exists a positive integer p such that Eq. (1) admits a solution expansion in the form

$$u=\sum_{0}^{\infty} u_j(x,t)\,\phi^{j-p}(x,t) \tag{5}$$

where $\phi(x,t)$ is not one of the characteristics of Eq. (1) and $u_j(x,t)$ are analytic functions in x and t. The positive integer p is found by an analysis of leading terms in (1) and is given by $p=n-1$. When substituting from (5) into Eq. (1), the resulting recursion relations must contain a sufficient number of positive resonance besides that one at $r=-1$. If the solution (5) contains a sufficient number of arbitrary functions $u_j(x,t)$ then Eq. (1) passes the Painlevé test. A truncated expansion of (5) is done by assuming that $u_{j+p}=0$, $j=1,2,\ldots$. Thus, the expansion (5) becomes

$$u=\sum_{j=0}^{p-1} u_j\phi^{j-p}+u_p \tag{6}$$

We substitute from Eq. (6) into Eq. (1). If the equations obtained for $j = 0, 1, \ldots$ are consistent, then (6) is an auto-Bäklund transformation for (1). From the consistency of these equations, we find the following set of equations.

$$u_p = G(\phi_t, \phi_x, \phi_{xx}, \ldots) \tag{7}$$

$$u_{pt} + u_p u_{px} + F(u_{pxx}, u_{pxxx}, \ldots) = 0 \tag{8}$$

$$G_0(\phi_t, \phi_x, \phi_{xx}, \ldots) = 0 \tag{9}$$

The last Eq. (9) is called the invariance condition. In general, Eqs. (7)–(9) form an overdetermined system of equations. But if Eq. (1) passes the Painlevé test, this system is not overdetermined. That is when substituting from (7) into (8), this gives rise to the Eq. (9) or to an equation derived from it by differentiation.[7] This result can not be proved for Eq. (1) in its implicit form and can be shown if (1) is given explicitly. A typical example is the KdV equation. It passes the Painlevé test and possesses an auto-Bäklund transformation. But the system (7)–(9) for this equation is not overdetermined. In this case the exact solution of (1) may be found by using the inverse scattering technique[9] or by using the Lie symmetries. If Eq. (1) does not pass the Painlevé test in the neighbourhood of arbitrary singular manifold ϕ, we may search for a special manifold where this test holds. In this case an auto-Bäklund transformation may exist. Thus, the system (7)–(9) is overdetermined and exact solutions of (1) may be fund by using the technique introduced in ref. 8.

Analysis of resonances of the KdV and the KS equations shows that they are at $r = -1, 4, 6$ and $r = -1, 4, 8$.[7, 10] By induction, the resonances of Eq. (1) are at $r = -1, 4, 2n$. On the other hand, the condition that an auto-Bäklund transformation for Eq. (1) exists is $r_{\max} - p - 1 \leqslant 3$ where $r_{\max}$ is the greatest positive integer. This condition reflects the fact that the number of the overdetermined equations in ϕ and u_p must not exceed three (cf. Eqs. (7)–(9)). Analysis of this condition shows that it holds when $n \leqslant 3$. It is known that an auto-Bäklund transformation does not exist for the KS equation ($n = 4$).[10] This drawback motivates one to search for other techniques to find special solutions of Eq. (1) for $n \geqslant 4$.

In the next section, we shall present a method that allows us to have an overdetermined system of equations. When solving this system, we can find some special solutions of (1). In Section (3), we apply this method to Eqs. (2)–(4). In Section (4) we discuss the common behavior of these solutions and give some conclusions and find the general form for a special solution of (1).

2. MATHEMATICAL FORMULATION

The idea, presented here, for finding an overdetermined system of equations is to search for special solutions which satisfy the superposition principle. Now, we assume that u_1 and u_2 are two solutions of (1), then we have

$$u_{1t} + u_1 u_{1x} + F(u_{1xx}, u_{1xxx}, \ldots) = 0 \tag{10}$$

$$u_{2t} + u_2 u_{2x} + F(u_{2xx}, u_{2xxx}, \ldots) = 0 \tag{11}$$

The condition that $u_1 + u_2$ is a solution of (1) is

$$u_1 u_{2x} + u_2 u_{1x} = 0 \tag{12}$$

The system of Eqs. (10)–(12) is overdetermined. Fortunately, Eq. (12) integrates to

$$u_2 = C(t)/u_1 \tag{13}$$

where $C(t)$ is an arbitrary function. By substituting from (13) into (11), we obtain

$$\frac{\dot{C}}{u_1} - \frac{Cu_{1t}}{u_1^2} - \frac{C^2 u_{1x}}{u_1^3} + CF\left(\frac{-u_{1xx}}{u_1^2} + \frac{2u_{1x}^2}{u_1^3}, \ldots\right) = 0 \tag{14}$$

By eliminating u_{1t} from (14), we have

$$\frac{\dot{C}}{u_1} + \frac{Cu_{1x}}{u_1} + \frac{C}{u_1^2} F(u_{1xx}, \ldots) - \frac{C^2 u_{1x}}{u_1^3} + CF\left(\frac{-u_{1xx}}{u_1^2} + \frac{2u_{1x}^2}{u_1^3}, \ldots\right) = 0 \tag{15}$$

Equation (15) contains only the spatial derivatives of the dependent variable u and may be integrable. The solution of (15) would contain a set of arbitrary functions in t. We insert this solution into (1) to find these arbitrary functions.

3. APPLICATIONS

3.1. The Burgers Equations

Here, we apply the technique presented in the previous section to the Eq. (2). We notice that by making a rescaling transformations, it is easy to see that we can confine ourselves to the cases $\lambda = 1$ and $\mu = \pm 1$. First, we consider the case $\lambda=1$ and $\mu=1$ and when substituting into (15), it becomes

$$\frac{\dot{C}}{u_1}-\frac{Cu_{1x}}{u_1}-\frac{C^2u_{1x}}{u_1^3}+\frac{2Cu_{1x}^2}{u_1^3}=0 \tag{16}$$

We notice hat Eq. (16) holds identically for $C=0$. When C is time independent, (16) integrates to either $u_1=const.$ or to

$$u_1=\begin{cases} C^{1/2}\tan chC^{1/2}[x/2+a(t)], & C>0 \\ |C|^{1/2}\tan|C|^{1/2}\,[a(t)-x/2], & C<0 \end{cases} \tag{17}$$

where $a(t)$ is an arbitrary function. When substituting from (17) into (2) we find that $a(t)=const.$ The two other solutions are

$$u_2=\begin{cases} C^{1/2}\coth C^{1/2}[x/2+a], & C>0 \\ -|C^{1/2}|\cot[-x/2+a], & C<0 \end{cases} \tag{18}$$

and u_1+u_2. When $\lambda=1$ and $\mu=-1$ in (2), calculations as in above give rise to the solutions to (17) and (18) but with x replaced by $-x$.

We remark that the solutions of Burgers equation which satisfy the superposition principle are the steady-state one. Also, the solutions which contain the tangent function explode in a finite value of x, namely $\sqrt{|C|}\,(a-x/2)\rightarrow\pm\pi/2$. Thus the solution behaves as $1/(x-x_o)$ in the neighbourhood of $x_o=2a-(\pm\pi/\sqrt{|C|})$. It is worth noticing that the traveling wave solution of (2) can be recasted into the steady state one by a suitable transformation.

Now, we assume that C in (16) is time dependent, it solves to

$$u_{1x}=\tfrac{1}{4}[(C-u_1^2)\pm\sqrt{u_1^4-2u_1^2(C+4\dot{C}/C)+C^2}] \tag{19}$$

Integration of (19) may be found. When substituting into (2), we find a trivial solution which is a special case of (23), namely when $C_o=0$. For this reason, details of the calculations are omitted.

We remark that the solutions of (2) which satisfies the superposition principle are the trivial ones. Now, we use the superposition principle in a modifies sense. For instance, we assume that u_2 satisfies (2) and u_1 satisfies only its dominant part, namely

$$u_1u_{1x}+u_{1xx}=0 \tag{20}$$

The condition that u_1+u_2 is a solution of (2) is

$$u_{1t}+(u_1u_2)_x=0 \tag{21}$$

By direct calculations, we find

$$u_1 = \begin{cases} 2\dfrac{C_o}{(b+t)} \tan ch \dfrac{C_o(x+a)}{b+t}, & C_o > 0 \\ 2\dfrac{|C_o|}{b+t} \tan \dfrac{|C_o|(-x+a)}{b+t}, & C_o < 0 \end{cases} \quad (22)$$

$$u_2 = \begin{cases} (x+a)/(t+b) + 8\dfrac{C_o^2}{(b+t)^2} \coth \dfrac{C_o(x+a)}{b+t}, & C_o > 0 \\ (x+a)/(t+b) - 8\dfrac{C_o^2}{(b+t)^2} \cot \dfrac{|C_o|(-x+a)}{b+t}, & C_o < 0 \\ (x+a)/(t+b), & C_o = 0 \end{cases} \quad (23)$$

where a and b are arbitrary constants. The solutions of Eq. (2) are u_2 and $u_1 + u_2$. We notice that these solution have not been found as classical similarity solutions of the Burgers equation.[11]

3.2. The KdV Equation

We apply the technique developed here to the Eq. (3) and consider only the cases $\lambda = 1$ and $\mu = \pm 1$ for the reasons mentioned previously. For $\lambda = 1$ and $\mu = 1$, Eq. (15) becomes

$$\frac{\dot{C}}{u_1} + \frac{Cu_{1x}}{u_1} - \frac{C^2}{u_1^3} - 6\frac{Cu_{1x}^3}{u_1^4} + 6\frac{Cu_{1x}u_{1xx}}{u_1^3} = 0 \quad (24)$$

Equation (24) holds identically for $C = 0$. Now, we assume that C is time independent, and (24) becomes

$$u_1^3 - Cu_1 - 6u_{1x}^2 + 6u_1 u_{1xx} = 0 \quad (25)$$

By using quadratures, Eq. (25) integrates to

$$\int^{u_1} \frac{dv}{\sqrt{-Cv - v^3 + C_o(t)\, v^2}} = \pm x/\sqrt{3} + C_1(t) \quad (26)$$

where $C_o(t)$ and $C_1(t)$ are arbitrary functions. Hereafter, we shall assume that C_o is a positive constant and we search for solutions of (3) which satisfies

$$u_1(0, t) = f(t) \quad (27)$$

In this case Eq. (26) becomes

$$\int_{f(t)}^{u_1} \frac{dv}{\sqrt{-Cv - v^3 + C_o v^2}} = \pm x/\sqrt{3} \tag{28}$$

When substituting from (28) into (3) we find that the function $f(t)$ satisfies the equation

$$\int_{f(0)}^{f(t)} \frac{dv}{\sqrt{v(a-v)(v-b)}} = \mp C_o t/3\sqrt{3} \tag{29}$$

where $a = C_o/2 + \sqrt{C_o^2/4 - C}$ and $b = C_o/2 - \sqrt{C_o^2/4 - C}$. Equation (29) shows that the function $f(t)$ is not arbitrarily chosen. When combining (29) and (26), we obtain the solution for (3). We have the following cases

(a) For $C > 0$ and $f(0) = b$, we find[12]

$$F(\chi, p) = \pm\sqrt{a/12}\,(x - C_o t/3) = z, \qquad \chi = \sin^{-1}\frac{a(u_1 - b)}{(a-b)\,u_1}, \qquad p = \sqrt{\frac{a-b}{a}} \tag{30}$$

where $F(x, y)$ is the elliptic integral of the first kind and $a > u_2 \geqslant b > 0$. Equation (30) solves to[13]

$$\chi = amz = \pi/2K(p) + \sum_{s=0}^{\infty} \frac{2q^s \sin(\pi z s/K(p))}{s(1+q^{2s})} \tag{31}$$

where $q = e^{-\pi K'(p')/K(p)}$ and $K(p)$ is the complete elliptic integral of the first kind. Finally we have

$$u_1 = \frac{ab}{a - (a-b)\, sn^2(z \mid p)} \tag{32}$$

where $sn(z \mid p)$ is the Jacobi elliptic function. The two other solutions are

$$u_2 = C/u_1, \qquad u = u_1 + u_2 \tag{33}$$

The results (32) and (33) are displayed in Fig. 1 for $C = 1$ and $C_o = 2\sqrt{2}$. The lower, middle and upper curves represent the solutions u_1, u_2, and $u_1 + u_2$ respectively. This figure shows the fact that the solutions (32), (33) are doubly periodic functions.

(b) For $C < 0$ and $f(0) = 0$, we find that the solution of (3) is

$$u_1 = acn^2(z^* \mid p_o) \tag{34}$$

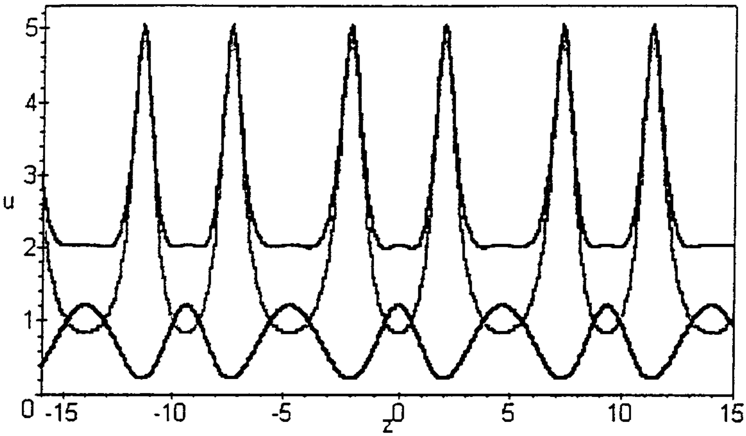

Fig. 1. Solutions given by the Eqs. (33) and (34). The lower, middle and upper curves correspond to the solutions u_1, u_2, and u_1+u_2 respectively.

where $z^* = \pm\sqrt{(a+b)/12}\,(x - C_o t/3)$, $p_o = \sqrt{(a+b)/a}$ and $a > u_1 > 0 > b$. The two other solutions are

$$u_2 = C/u_1, \qquad u = u_1 + u_2 \tag{35}$$

The solutions given by (34, 35) are displayed in Fig. 2 for $C = -1$ and $C_o = 2\sqrt{2}$. The upper, middle and lower curves represent the solutions u_1, u_2, and $u_1 + u_2$ respectively. After this figure, the solutions u_2 and $u_2 + u_1 \to -\infty$ for finite values of z^*. This occurs in a doubly periodic manner.

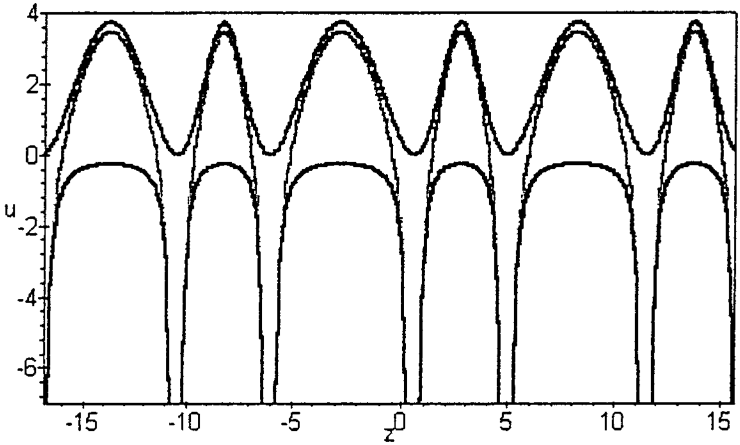

Fig. 2. Solutions given by the Eqs. (35, 36). The upper, middle and lower curves correspond to the solutions u_1, u_2, and u_1+u_2 respectively.

(c) For $C=0$ and $f(0)=0$, we find that

$$u = C_o \sec h^2 \sqrt{\frac{C_o}{3}}\,(x - C_o t/3) \tag{36}$$

This solution for the KdV equation is known as the single soliton solution.

Now, we turn to the case when $\lambda=1$ and $\mu=-1$ in (3). When $C=0$ and $f(0)=0$, we find that

$$u_1 = \begin{cases} C_o \sec h^2 \sqrt{\frac{C_o}{3}}\,(C_o t/3 - x), & C_o > 0 \\ |C_o| \sec^2 \sqrt{\frac{|C_o|}{3}}\,(x + |C_o t|/3), & C_o < 0 \end{cases} \tag{37}$$

From (37), and for $C_o < 0$ we remark that the solution of the KdV explodes as $z + \sqrt{|C_o|/3(x + |C_o|\,t/3)} \to \pm\pi/2$. Thus, the solution given by (37) explodes whenever the variables x and t lie on the characteristics $(x + |C_o|\,t/3) = pm\pi\sqrt{3/4\,|C_o|}$. This behavior for the solution of the KdV equation holds only when $\lambda=1$ and $\mu=-1$. It does not hold when $\lambda=1$ and $\mu=1$.

In (24) when C is time dependent, by using the transformation $u_x = hu$, it becomes an Abel equation on the first kind.

3.3. The KS Equation

It has been shown that the KS equation is not integrable.[14] It does not pass the Painlevé test and an auto-Bäklund transformation for this equation does not exist.[10] Consequently, no special solutions are found by using the technique of Conte.[8]

We apply the technique presented in Section (2) to the Eq. (4) for $\lambda=\mu=\nu=1$. In this case, Eq. (15) becomes

$$\frac{\dot{C}}{u_1} + \frac{Cu_{1x}}{u_1} - \frac{C^2 u_{1x}}{u_1^3} + \frac{2Cu_{1x}^2}{u_1^3} + \frac{8Cu_{1x}u_{1xxx}}{u_1^3} + 6\frac{Cu_{1xx}^2}{u_1^3} - 36\frac{Cu_{1x}^2 u_{1xx}}{u_1^4} + 24\frac{Cu_{1x}^4}{u_1^5} = 0 \tag{38}$$

Unfortunately, it is difficult to solve Eq. (38) exactly even though $C = const$. We use the "modified" superposition principle and assume that u_1 is a solution of the equation

$$u_{1xx} + u_1 u_{1x} + u_{1xxxx} = 0 \tag{39}$$

and u_2 satisfies Eq. (4). The condition that u_1+u_2 satisfies also (4) is given by (21). After the results found in this section, one can inspect easily as special solution of (39). We have found that if the term u_{xx} (or u_{xxx}) is present in (1), then its solution will contain the term tanhz (tanz) (or $\tanh^2 z$ $(\tan^2 z)$) respectively. This may bee seen from the solutions of the Burgers and KdV equations. Thus, as u_{xxxx} is present in (39), then its solution will contain $\tanh^3 z$ $(\tan^3 z)$. Here, z may designate the characteristic variable (or $z=C(t)\,x+D(t)$). By inspection, the presence of a linear combination of the spatial derivative terms give rise to a linear combination of $\tanh^m z$ $(\tan^m z)$. This suggests to writing the solution of Eq. (39) in the form

$$u_1 = A(t)\tanh(C(t)\,x+D(t)) + B\tanh^3(C(t)\,x+D(t)) \tag{40}$$

where A, B, C, and D are arbitrary functions in t. The solution (40) satisfies the integrated form of (39), namely

$$u_{xxx}+\frac{u^2}{2}+u_x+K(t)=0 \tag{41}$$

where $K(t)$ is an arbitrary function in t, when the following equations hold

$$\begin{aligned} B^2-120BC^3&=0\\ 6C^3(9B-1)+60BC^3-3BC+AB&=0 \qquad (42)\\ C^3(8A-60B)-CA+3BC+\frac{A^2}{2}&=0\\ 2C^3(2B-A)+AC+K&=0 \qquad (43)\end{aligned}$$

Equations (42) and (43) solve to

$$A=\frac{60}{57}\,C(3-114C^2), \quad B=120C^3, \quad C^2=11/76, \quad C^2=-1/76 \tag{44}$$

The condition that u_1+u_2 is a solution of (4) is

$$u_{1t}+(u_1u_2)_x=0 \tag{45}$$

where u_2 satisfies (4). By taking $u_2=D_o=\text{const}$ and substituting from (40), (41) into (42) we obtain

$$\dot{C}+D_oC=0 \tag{46}$$

Finally, we have a solution for (4) which is given by

$$u = u_1 + u_2 = D_o + u_1 \tag{47}$$

where in (40), $D(t) = -CD_o t + D_1$; D_o and D_1 are constants.

When $D_o = D_1 = 0$, the solutions (40), (44), and (47) becomes the Kuramoto–Tsuzuki solution.[15] When $C^2 = -1/76$, this solution becomes

$$u = \frac{-5}{19^{3/2}} (3 \tan |k|\, x + \tan^3 |k|\, x), \qquad |k| = 1/\sqrt{76} \tag{48}$$

Now, after (48), the solution of the KS equation blows up to infinity as $|k|\, x \to \pm \pi/2$. Thus, by maintaining only the dominant term in (48), we find that $u \approx (x - x_o)^{-3}$ where $x_o = \pi \sqrt{19}$. This result holds also when $(\lambda, \mu, \nu) = (1, 1, -1)$.[16] After the results of this section, we conjecture that

(i) In Eq. (1), if the maximum order of the spatial derivative term is even, then it possesses a solution which explodes in a finite value for the spatial (or the characteristic) variable whatever the coefficients λ, μ, ν,... etc. But if the maximum order is odd then the solution explodes only if the coefficients of the convective and spatial derivative terms are of opposite signs.

(ii) The power of "explosion" depends on the maximum order of the spatial derivative term in (1).

We shall prove these conjectures in the next section.

4. DISCUSSIONS AND CONCLUSIONS

We have presented a method that allows us to find some special solutions of Eq. (1). New special solutions for the Burgers and KdV equations have been found. We have also shown that (1) possesses a solution which explodes in a finite value of the spatial or the characteristic variable. This occurs according to the statements (i) and (ii) in the above. We have demonstrated that these two statements hold for the Burgers, the KdV and the KS equations. Here, we shall prove these two statements in the general case for (1). To this end, we consider the canonical forms of Eq. (1) when n is even and odd respectively,

$$u_t + \lambda_1 u u_x + \mu_1 \frac{\partial^{2m} u}{\partial x^{2m}} = 0 \tag{49}$$

$$u_t + \lambda_2 u u_x + \mu_2 \frac{\partial^{2m+1} u}{\partial x^{2m+1}} = 0 \tag{50}$$

Now, we search for the traveling wave solution with characteristic variable $z = x - ct$ for Eqs. (45) and (46) and they become

$$-cu' + \lambda_1 uu' + \mu_1 u^{(2m)} = 0 \tag{51}$$

$$-cu' + \lambda_2 uu' + \mu_2 u^{(2m+1)} = 0 \tag{52}$$

where the prime denotes the derivative with respect to z. Equations (47) and (48) are integrated to

$$-cu + \frac{\lambda_1}{2}u^2 + \mu_1 u^{(2m+1)} + K_1 = 0 \tag{53}$$

$$-cu + \frac{\lambda_2}{2}u^2 + \mu_2 u^{(2m)} + K_2 = 0 \tag{54}$$

By making the transformation $(u - c/\lambda_j) \to u$ and if we choose $K_j - c^2/\lambda_1^2 = 0$, then Eqs. (53) and (54) become

$$\frac{\lambda_1}{2}u^2 + \mu_1 u^{(2m-1)} = 0 \tag{55}$$

$$\frac{\lambda_2}{2}u^2 + \mu_2 u^{(2m)} = 0 \tag{56}$$

If Eqs. (55) and (56) admit particular solutions in the form $u' = \alpha u^r$, $r > 1$, then these solutions explode in a finite value of z. In order that these solutions existed, two conditions must hold; $r > 1$ and α is real. In fact, $r = 2m/(2m-1)$ and $r = 1 + 1/2m$ corresponding to Eqs. (55) and (56) respectively. For the second condition, we require that the solution of the following two equations in α is real;

$$\frac{\lambda_1}{2} + \mu_1(2r-1)(3r-2)\cdots(2m(r-1)+3)\,\alpha^{2m-1} = 0 \tag{57}$$

$$\frac{\lambda_2}{2} + \mu_2(2r-1)(3r-2)\cdots((2m+1)\,r+3)\,\alpha^{2m} = 0 \tag{58}$$

Equation (57) solves to a real value of α whatever the signs of λ_1 and μ_1. But a solution of (58) for α exists only if λ_2 and μ_2 have opposite signs. This proves the first statement. To show that the second statement holds, we assume that the two conditions mentioned above are satisfied. That is $u' = \alpha u^r$, $r > 1$. Thus, we find that Eqs. (55) and (56) have special solutions which explode as $(\beta_1 - (\alpha_1/2m-1)\, z)^{2m-1}$ and $(\beta_2 - (\alpha_2/2m)\, z)^{2m}$ respectively. The statement (i) in above is not surprising because in (49) we can always make λ_1 and μ_1 have the same sign by the transformation $x \to -x$. But this is not the case for Eq. (50).

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